A Weil Descent Attack against Elliptic Curve Cryptosystems over Quartic Extension Fields (Algebraic Aspects of Coding Theory and Cryptography)

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A Weil Descent Attack against Elliptic Curve Cryptosystems over Quartic Extension Fields

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1. Introduction

離散対数型暗号

G:有限アーベル群
G ∋ g を固定

指数関数E

$$E: Z \to G$$
$$n \mapsto g^n$$

離散対数問題

<g> ∋h に対して E(n) = h となる n を求めよ

離散対数問題が困難ならば,
群Gを用いて公開鍵暗号が構成できる(ElGamal)

超楕円曲線暗号

GF(q)上の種数gの超楕円曲線 H （$g \ll q$）

$$y^2 = f(x) = x^{2g+1} + a_1 x^{2g} + \cdots + a_{2g+1} \quad (a_i \in GF(q))$$

超楕円曲線Hのヤコビ群 J_H

$$J_H = \{ (u(x),\ v(x)) \mid \deg v < \deg u \leqq g,\ v^2 = f \pmod u \}$$
$$\# J_H \fallingdotseq q^g$$

超楕円曲線H上の離散対数問題(DLP)

$J_H \ni D_1, D_2$ に対して $D_2 = n D_1$ となるnを求めよ

Pollardのrhoアルゴリズム

$J_H \ni D_1, D_2$

random walk によって
D_1, D_2のランダムな線形和R_i を計算していく

$$R_i = \alpha_i D_1 + \beta_i D_2 \quad (i=1,2,3,\ldots)$$

衝突$R_i = R_j$ が起きると

$$\alpha_i D_1 + \beta_i D_2 = \alpha_j D_1 + \beta_j D_2$$

$$\therefore\ D_2 = (\alpha_i - \alpha_j) / (\beta_j - \beta_i)\ D_1$$

$(\#J_H)^{1/2}$ step が必要

Gaudry's 法

$J_H \ni D$ が smooth
Dが定義体上の有理点の和

$FB = \{ P_1, P_2, \ldots, P_w \}$: 超楕円曲線H上の全ての有理点を計算しておく ($w \fallingdotseq q$, $O(q)$)

$J_H \ni D_1, D_2$

random walk によって D_1, D_2のランダムな線形和R_i を計算:

$$R_i = \alpha_i D_1 + \beta_i D_2 \quad (i=1,2,3,\ldots)$$

smooth な R_i を集める

Gaudry's 法(2)

smooth な R_i がみつかると: (g! 個に一つは smooth)

$R_i = (u(x), v(x))$
$u(x) = \prod (x - x_i)$: GF(q)上1次の既約因子に分解 ($O(q)$)
$y_i = v(x_i) \quad (i=1,\ldots,g)$

$$R_i = (x_1,y_1) + (x_2,y_2) + \ldots + (x_g, y_g)$$

$$= \Sigma_k \, m_{i,k} P_k$$

このようにして

$$\text{smoothな} R_i \longleftrightarrow [m_{i,1}, m_{i,2}, \ldots, m_{i,w}]$$

Gaudry's 法(3)

smooth な $R_i = \Sigma_k\ m_{i,k} P_k$ が w'(>w) 個あつまると ($O(q^2)$) :

$$M := (m_{i,k}) \quad : w' \times w \text{ 行列, 疎}$$

$(\gamma_i) \in \mathrm{Ker}\ {}^tM$ を求めて ($O(q^2)$)

$$\Sigma_i\ \gamma_i R_i = 0$$

$$\therefore\ \Sigma_i\ \gamma_i (\alpha_i D_1 + \beta_i D_2) = 0$$

$D_2 = \lambda\ D_1$ を 代入すると

$$\lambda = (-\ \Sigma\ \gamma_i\ \alpha_i)(\Sigma\ \gamma_i\ \beta_i)^{-1}$$

Gaudry法

超楕円曲線上のDLPに対する攻撃法
rho アルゴリズムに因子基底を導入
$O(q^2 \log^c(q))$ (定義体: GF(q), 種数g ≪ q)

→ 種数を大きくしても, q は小さくできない！

C_{ab}曲線や superelliptic 曲線に対しても
Gaudry's 法 は 有効

C_{ab}曲線と superelliptic 曲線

C_{ab}曲線: (a,b) = 1

$$\sum_{0\le i\le b,0\le j\le a,ai+bj\le ab} \alpha_{i,j} x^i y^j = 0$$

superelliptic 曲線: (n, δ) = 1

$$y^n = a_\delta x^\delta + ... + a_0$$

明らかに,

superelliptic 曲線 $\subset$ C_{ab}曲線

実装実験 — パラメータ

有限体	GF(84211)
定義方程式	1 + 24740 x^7 + 32427 y^3 = 0
種数	6
ヤコビアンの位数	43・806897062301623960531898661 7
自己同型の位数	3・7

17ビットの素体上の93ビットC_{37}

自己同型 ϕ

$\phi(x, y) = (\zeta_7 x, \zeta_3 y)$: 位数21

よって

#FB = 84211 / 21 = 4010.・・

実装実験 — 結果

有理点の収集 (PARI-GP)	5分13秒
スムースな要素の収集 (C)	2時間33分6秒
一次方程式の求解 (C)	32分2秒
合計	3時間11分21秒

17ビットの素体上の93ビットC_{37}曲線に対するGaudry's variant の適用,
266 MHz Pentium II

Weil descent attack

$K = \mathrm{F}_{q^n}$ / $k = \mathrm{F}_q$
H: an algebraic curve over K

geometrically defined over k

$\Pi_{K/k}\, H \supset D \overset{\sim}{\rightleftharpoons} C$: a model over k

Problem 1
How to choose D of small genus?

π

H

Φ

Problem 2
How to construct C with computable Jacobian ?

Gaudry attack

$\mathrm{N}_{K|k} \cdot \Phi^* : J_K(H) \longrightarrow J_k(C)$

C has larger genus and smaller field than H.

Weil descent attack
possibly applied against
an algebraic curve cryptosystem over a composition field

The concept: Frey '98
(some of) ECC over char. 2 finite fields: Gaudry,Hess,Smart '00
(some of) HCC over char. 2 finite fields: Galbraith '00
(some of) ECC over char. 3 finite fields : Arita '00
(some of) ECC, HCC over odd char. finite fields : Diem '00

Our contributions

k=GF(q), ch(q)$\neq$2,3
k_2: quadratic ext. of k, k_4: quartic ext. of k
E: elliptic curve over k_4

$$E: v^2 = u^3 + \alpha u + \beta \qquad (\alpha, \beta \in k_4)$$

Π ↑

$$H: y^2 = x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f \qquad (a,b,c,d,e,f \in k_2)$$

$\Longleftarrow$ GHS attack

- In this talk, we show that
  many elliptic curve cryptosystems over quartic degree finite fields
  come under Weil descent attack.

2. Scholten form

Scholten form

k=GF(q) (ch(k) ≠ 2,3)
k_4: quartic ext. of k

E_n: Scholten form (of elliptic curve) over k_4

$$v^2 = \alpha u^3 + \beta u^2 + \beta^{q^2} u + \alpha^{q^2} \qquad (\alpha, \beta \in k_4)$$

Scholten showed

- $\Pi_{k4|k2} E_n \sim J_{k2}(H)$ ($\exists$ H: genus 2 HEC)
- E/k_4 has full 2-torsions $\Rightarrow$ E has Scholten form

We clarify conditions to be Scholten form.

Scholten form is covered by H

$$E_n: v^2 = \alpha u^3 + \beta u^2 + \beta^{q2} u + \alpha^{q2} \quad (\alpha, \beta \in k_4)$$

$$\uparrow \quad (u, v) = \Pi((x-c)^2/(x-c^{q^2})^2,\ y/(x-c^{q^2})^3) \quad (c \in k_4\text{-}k_2)$$

$$H: y^2 = \alpha(x-c)^6 + \beta(x-c)^4(x-c^{q^2})^2 + \beta^{q^2}(x-c)^2(x-c^{q^2})^4 + \alpha^{q^2}(x-c^{q^2})^6$$

- H: defined over k_2
- DLP on E_n/k_4 → DLP on H/k_2

When E_w can be in Scholten form ?

$E_w : y^2 = f(x)$: Weierstrass form / k_4, $f(x)$: irreducible / k_4

$$\xrightarrow[x \to Ax + B,\ y \to Cy \quad (A,B,C \in k_4)]{} \quad E_n: y^2 = F(x): \text{Scholten form / } k_4$$

δ : a root of $F(x) = a x^3 + b x^2 + b^{q2} x + a^{q2}$

$$\delta^{-q2} \in \{ \delta, \delta^{q4}, \delta^{q8} \}$$

$$\delta^{-q2} = \delta \Rightarrow \delta^{q4-1} = 1 \Rightarrow \delta \in k_4$$
$$\delta^{-q2} = \delta^{q4} \Rightarrow \delta^{q2} = \delta^{-1} \Rightarrow \delta \in k_4$$

$$\therefore \quad \delta^{1+q6} = 1$$

Proposition 2

$E_w : y^2 = f(x)$: Weierstrass form / k_4, $f(x)$: irreducible / k_4

E_w is isomorphic to Scholten form E_n over k_4

$\Leftrightarrow$

$(*)$ $\quad \exists A(\neq 0) \in k_4,\ B \in k_4$

$\gamma = A\delta + B,\quad \delta^{1+q6} = 1 \quad (\gamma$: a root of $f(x))$

Then,

$$a := -A^{2-q2}\,\delta^{1+q4-q2},\quad b := -A(\delta + \delta^{q4} + \delta^{q-2})$$

$$E_w \to E_n : y^2 = a\,x^3 + b\,x^2 + b^{q2}\,x + a^{q2}$$

$$(\,y \to a\,y,\ x \to a\,x + B\,)$$

$f(x)$: irreducible / k_4, γ : a root of $f(x)$

$$d(\gamma) := (\gamma^{q2+q4} - \gamma^{q2+1}) + (\gamma^{q6+q8} - \gamma^{q6+q4}) + (\gamma^{q10+1} - \gamma^{q10+q8})$$

$$d(\gamma) \neq 0$$

$\Leftrightarrow$

$(*)$ $\quad \exists A(\neq 0) \in k_4,\ B \in k_4$

$\gamma = A\delta + B,\quad \delta^{1+q6} = 1 \quad (\gamma$: a root of $f(x))$

$\because$

$$(\gamma - B)^{1+q6} \in k_2$$

$$\Leftrightarrow (\gamma - B)^{q2}\,(\gamma^{q6} - B^{q6})^{q2} = (\gamma - B)(\gamma^{q6} - B^{q6})$$

$$\Leftrightarrow \begin{cases} g(B) = 0 \\ g^{q2}(B^{q2}) = 0 \end{cases}$$

Bの連立1次方程式

E_w: $y^2 = f(x)$ with $f(x)$: irreducible / k_4, γ : a root of $f(x)$

$$d(\gamma) = (\gamma^{q2+q4} - \gamma^{q2+1}) + (\gamma^{q6+q8} - \gamma^{q6+q4}) + (\gamma^{q10+1} - \gamma^{q10+q8})$$

$$d(\gamma) = 0 \Leftrightarrow j(E_w) \in k_2$$

$\because$

$$j(E_w) \in k_2 \Leftrightarrow \gamma = A\alpha + B \ (A, B \in k_4, \ \alpha \in k_6)$$
$$\Leftrightarrow d(\gamma - B) = 0$$
$$\Leftrightarrow d(\gamma) = 0$$

When E_w can be in Scholten form

E_w : $y^2 = f(x)$: Weierstrass form / k_4

- $f(x)$: 既約 / k_4
  $j(E_w) \in k_4 - k_2$ ならば、
  E_wはScholten formに k_4上で変換される。

- $f(x)$ = 1次式 x 既約2次式 / k_4
  E_wはScholten formで表されない。

- $f(x)$: 完全分解 / k_4
  E_wは常にScholten formに k_4上で変換される。

3. GHS attack for genus 2 HEC

GHS attack in our case

k_2=$\mathrm{F}_q{}^2 \mid k = \mathrm{F}_q$ (of char.$\neq 2$), σ : Frob. Automorphism of k_2/k
H: $y^2 = x^6 + a\,x^5 + b\,x^4 + c\,x^3 + d\,x^2 + e\,x + f$ $(a,b,c,d,e,f \in k_2)$

$\sigma : \mathrm{x}_1 \rightarrow \mathrm{x}_2, \mathrm{y}_1 \rightarrow \mathrm{y}_2$

$$\Pi_{\mathrm{k2/k}}\, H : \begin{cases} y_1{}^2 = x_1{}^6 + a\,x_1{}^5 + b\,x_1{}^4 + c\,x_1{}^3 + d\,x_1{}^2 + e\,x_1 + f \\ y_2{}^2 = x_2{}^6 + a^q x_2{}^5 + b^q x_2{}^4 + c^q x_2{}^3 + d^q x_2{}^2 + e^q x_2 + f^q \end{cases}$$

$$\cup \qquad x := x_1 = x_2$$

GHS-section D:

$$\begin{cases} y_1{}^2 = x^6 + a\,x^5 + b\,x^4 + c\,x^3 + d\,x^2 + e\,x + f \\ y_2{}^2 = x^6 + a^q x^5 + b^q x^4 + c^q x^3 + d^q x^2 + e^q x + f^q \end{cases}$$

Assumption for non-singularity

$H: y^2 = x^6 + a x^5 + b x^4 + c x^3 + d x^2 + e x + f$ on $k_2 = F_q^2 / k = F_q$

Assumption

$x^6 + a x^5 + b x^4 + c x^3 + d x^2 + e x + f$ contains no non-trivial factor defined over k.

↓ then

GHS-section D is non-singular as affine curve.

Genus of GHS-section *D*

GHS-section D:

$$\begin{cases} y_1^2 = x^6 + a x^5 + b x^4 + c x^3 + d x^2 + e x + f \\ y_2^2 = x^6 + a^q x^5 + b^q x^4 + c^q x^3 + d^q x^2 + e^q x + f^q \end{cases}$$

points with $y_1=0$

↓ 12 ramification points

$H: y^2 = x^6 + a x^5 + b x^4 + c x^3 + d x^2 + e x + f$

Hurwitz formula

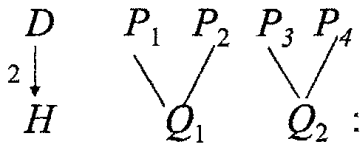
$$2g(D)-2 = [K(D):K(H)] \cdot (2g(H)-2) + \Sigma (e(P'|P)-1) \cdot \deg P'$$

$\therefore$ genus of GHS-section D = 9; small ⟹ Problem 1

What we need to do is to construct C_{ab} model over k of GHS-section D. ⟹ Problem 2

Points at infinity of GHS-section D

$\Pi_{k2|k}H \supset D$: $x=x_1=x_2$; GHS-section



$v_{Qi}(x)=-1$, $v_{Qi}(y)=-3$

: two points at infinity of H

$t = x^2/y_1$: local parameter at Q_1 and Q_2

$$\begin{cases} x = t^{-1} + \alpha_0^{(i)} + \alpha_2^{(i)} t + \cdots \\ y_1 = t^{-3} + \beta_{-2}^{(i)} t^{-2} + \beta_{-1}^{(i)} t^{-1} + \cdots \end{cases} \quad \text{at } Q_i \text{ (i=1,2)}$$

Substituting the above x for the second equation $y_2^2 = x^6 + a^q x^5 + b^q x^4 + c^q x^3 + d^q x^2 + e^q x + f^q$ of GHS-section D,

$$y_2 = -t^{-3} + \gamma_{-2}^{(2i-1)} t^{-2} + \cdots \quad \text{at } P_{2i-1} \text{ (i=1,2)}$$
$$\text{or} \quad y_2 = t^{-3} + \gamma_{-2}^{(2i)} t^{-2} + \cdots \quad \text{at } P_{2i} \text{ (i=1,2)}$$

Points at infinity of GHS-section

$\Pi_{k2|k}H \supset D$: $x=x_1=x_2$ $\quad t = x^2/y$

$$P_1 \begin{cases} x = -t^{-1} + \alpha_0^{(1)} + \alpha_1^{(1)} t + \cdots \\ y_1 = t^{-3} + \beta_{-2}^{(1)} t^{-2} + \beta_{-1}^{(1)} t^{-1} + \cdots \\ y_2 = -t^{-3} + \gamma_{-2}^{(1)} t^{-2} + \gamma_{-1}^{(1)} t^{-1} + \cdots \end{cases}$$

$$P_2 \begin{cases} x = -t^{-1} + \alpha_0^{(1)} + \alpha_1^{(1)} t + \cdots \\ y_1 = t^{-3} + \beta_{-2}^{(1)} t^{-2} + \beta_{-1}^{(1)} t^{-1} + \cdots \\ y_2 = t^{-3} + \gamma_{-2}^{(2)} t^{-2} + \gamma_{-1}^{(2)} t^{-1} + \cdots \end{cases}$$

$$P_3 \begin{cases} x = t^{-1} + \alpha_0^{(2)} + \alpha_1^{(2)} t + \cdots \\ y_1 = t^{-3} + \beta_{-2}^{(2)} t^{-2} + \beta_{-1}^{(2)} t^{-1} + \cdots \\ y_2 = -t^{-3} + \gamma_{-2}^{(3)} t^{-2} + \gamma_{-1}^{(3)} t^{-1} + \cdots \end{cases}$$

$$P_4 \begin{cases} x = t^{-1} + \alpha_0^{(2)} + \alpha_1^{(2)} t + \cdots \\ y_1 = t^{-3} + \beta_{-2}^{(2)} t^{-2} + \beta_{-1}^{(2)} t^{-1} + \cdots \\ y_2 = t^{-3} + \gamma_{-2}^{(4)} t^{-2} + \gamma_{-1}^{(4)} t^{-1} + \cdots \end{cases}$$

⟶ 'value' of any polynomial $f(x,y_1,y_2)$ at P_i, $\sigma(P_4) = P_4$

C_{ab} model of GHS-section D

- We construct C_{ab} model of D with the point P_4 at infinity as a base point.

Suppose P_4 is not a Weierstrass point:

Pole numbers at $P_4 = < 10,11,\ldots,19 >$.

Construct
a polynomial f_i with a unique pole of order i at P_4
for $i = 10,11, \ldots, 19$.

Note: t-expansion of P_i gives the value at P_i

$f_{10}, f_{11}, \ldots, f_{19} \longrightarrow$ $C_{10,11,\ldots,19}$ model over k_2 of D

$g_i := \mathrm{Tr}_{k2|k}(f_i)$ $\quad (\mathrm{Tr}_{k2|k}(f) = f + \sigma(f))$

$g_{10}, g_{11}, \ldots, g_{19} \longrightarrow$ $C_{10,11,\ldots,19}$ model C over k of D

Note: P_4 is fixed by σ.

Reduction : H → C

$$\begin{array}{ccc} D & \xrightarrow{g=(g_{10},g_{11},\ldots,g_{19})} & C\ /k_2 \\ \pi \downarrow & \swarrow \Pi_1 & \\ H & & \end{array}$$

$$\pi^{-1}(S_1) + \pi^{-1}(S_2) - (P_1+P_2+P_3+P_4) \sim \Sigma R_i - n P_4 \xrightarrow{g^*} \Sigma g(R_i) - n\infty$$

$$\pi^* \uparrow \qquad\qquad \nearrow \Pi_1^*$$

$$h = S_1 + S_2 - (Q_1+Q_2) \in J_{k2}(H)$$

Reduction: H → C with ideals

$$\begin{array}{ccc} k_2(D)=k_2(x,y_1,y_2) & \xrightarrow{g^*} & k_2(g_{10},g_{11},\ldots,g_{19}) \\ \pi^* \uparrow & \nearrow \Pi^* & \\ k_2(H)=k_2(x,y_1) & & \end{array}$$

$w := h \cdot g$
$\subset k_2[x,y_1,y_2]$ ⟶ an ideal $v \subset k_2[g_{10},g_{11},\ldots,g_{19}]$ of relations among $g_{10}(x,y_1,y_2), g_{11}(x,y_1,y_2), \ldots, g_{19}(x,y_1,y_2)$ mod w

$(v_{P_i}(g) \geqq 1, i = 1,2,3)$ ↑

an ideal $h \subset k_2[x,y_1]$

$v = g^*(w) = \Pi^*(h)$

Reduction: E → C

C: C_{ab} model over k of GHS-section D

$$\begin{array}{ccc} C & \xrightarrow{\sim} & D \\ (g_{10},g_{11},\ldots,g_{19}) & \xrightarrow{\phi} & (x,y_1,y_2) \\ & & \pi \downarrow \quad \downarrow \\ & \Pi_1 \searrow & H \quad (x,y_1) \\ \Pi \searrow & & \Pi_2 \downarrow \\ & & E \end{array}$$

$$\Psi : E(k_4) \xrightarrow{\Pi^*} \mathrm{Jac}_C(k_4) \xrightarrow{N_{k_4/k}} \mathrm{Jac}_C(k)$$

Ψ reduces DLP on E/k_4 to DLP on C/k.

Example1 : Transformation to palindrome form

k: prime field of char. $q=p=71$
k_2: quadratic ext. by o^2-2o+7 of k
k_4: quartic ext. by r^2-or+1

$\#E(k_4)=n=25404727$:prime
$j(E_w)=o^{1854}\,r+o^{2692} \notin k_2$

$$E_w/k_4: v_1^2 + 70\,u_1^3 + (o^{2058}\,r + o^{4231})\,u_1 + o^{3375}\,r + o^{2069} = 0$$

$$\wr \quad \Pi_2^{(1)}: \begin{cases} u = a^{-1}(u_1 + \beta_2) \\ v = a^{-1} v_1 \end{cases}$$

$a=o^{2258}\,r + o^{214}$
$b=o^{3519}\,r + o^{2654}$
$\beta_2=o^{4167}\,r + o^{3302}$

$$E_n/k_4: v^2 = a\,u^3 + b\,u^2 + b^{q2}\,u + a^{q2}$$
$$= (o^{2258}r + o^{214})\,u^3 + (o^{3519}\,r + o^{2654})\,u^2 + (o^{999}r + o^{3103})\,u + o^{4778}r + o^{355}$$

Example1 : Covering by hyperelliptic curve

$$E_n/k_4: v^2 = a\,u^3 + b\,u^2 + b^{q2}\,u + a^{q2}$$
$$= (o^{2258}r + o^{214})\,u^3 + (o^{3519}\,r + o^{2654})\,u^2 + (o^{999}r + o^{3103})\,u + o^{4778}r + o^{355}$$

$$\uparrow \quad \Pi_2^{(2)}: \begin{cases} u = (x_0 - c)/(x_0 - c^{q2})^2 \\ v = y_0/(x_0 - c^{q2})^3 \end{cases}$$

$$H_0/k_2: y_0^2 = a(x_0-c)^6 + b(x_0-c)^4(x_0-c^{q2})^2 + b^{q2}(x_0-c)^2(x_0-c^{q2})^4 + a^{q2}(x_0-c^{q2})^6$$
$$= o^{1463}x_0^6 + o^{666}x_0^5 + o^{2070}x_0^4 + o^{1093}x_0^3 + o^{794}x_0^2 + o^{315}x_0 + o^{1939}$$

$$\wr \quad \Pi_2^{(3)}: \begin{cases} y_1 = F(\beta)^{-(1/2)}(x_0 - \beta)^{-3} y_0 \\ x = 1/(x_0 - \beta) \end{cases}$$

$c = r$
$H_0: y_0^2 = F(x)$
$\beta = 3$

$$H/k_2: y_1^2 = x^6 + o^{2177}x^5 + o^{4311}x^4 + o^{2447}x^3 + o^{566}x^2 + o^{3664}x + o^{3747}$$

$$\Pi_2 := \Pi_2^{(1)}\,\Pi_2^{(2)}\,\Pi_2^{(3)}: H \longrightarrow E_w$$

Example1 : Points at infinity of GHS-section *D*

$\Pi_{k2|k}H \supset D$: $x=x_1=x_2 \quad t=x^2/y$

$$P_1 \begin{cases} x = 70\,t^{-1} + o^{4265} + o^{261}\,t + o^{4535}\,t^2 + o^{2836}\,t^3 + \cdots \\ y_1 = t^{-3} + o^{2177}\,t^{-2} + o^{4111}\,t^{-1} + o^{3867} + o^{3086}\,t + \cdots \\ y_2 = 70\,t^{-3} + o^{2713}\,t^{-2} + o^{4163}\,t^{-1} + o^{3058} + o^{4299}\,t + \cdots \end{cases}$$

$$P_2 \begin{cases} x = 70\,t^{-1} + o^{4265} + o^{261}\,t + o^{4535}\,t^2 + o^{2836}\,t^3 + \cdots \\ y_1 = t^{-3} + o^{2177}\,t^{-2} + o^{4111}\,t^{-1} + o^{3867} + o^{3086}\,t + \cdots \\ y_2 = t^{-3} + o^{193}\,t^{-2} + o^{1643}\,t^{-1} + o^{538} + o^{1779}\,t + \cdots \end{cases}$$

$$P_3 \begin{cases} x = t^{-1} + o^{4265} + o^{2781}\,t + o^{4535}\,t^2 + o^{316}\,t^3 + \cdots \\ y_1 = t^{-3} + o^{4697}\,t^{-2} + o^{4111}\,t^{-1} + o^{1347} + o^{3086}\,t + \cdots \\ y_2 = 70\,t^{-3} + o^{193}\,t^{-2} + o^{4163}\,t^{-1} + o^{538} + o^{4299}\,t + \cdots \end{cases}$$

$$P_4 \begin{cases} x = t^{-1} + o^{4265} + o^{2781}\,t + o^{4535}\,t^2 + o^{316}\,t^3 + \cdots \\ y_1 = t^{-3} + o^{4697}\,t^{-2} + o^{4111}\,t^{-1} + o^{1347} + o^{3086}\,t + \cdots \\ y_2 = t^{-3} + o^{2713}\,t^{-2} + o^{1643}\,t^{-1} + o^{3058} + o^{1779}\,t + \cdots \end{cases}$$

⟶ 'value' of any polynomial $f(x,y_1,y_2)$ at P_i

Example1 : C_{ab} model of GHS-section *D*

a polynomial with a unique pole of order i at P_4 : $g_i = \mathrm{Tr}_{k2|k}(f_i)$

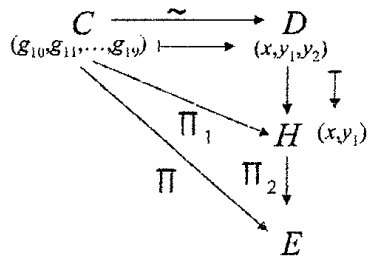
$$\begin{cases} g_{10} = o^{1264}\,x^3\,y_1^2 + 3\,x^3\,y_1\,y_2 + o^{271}\,x^3\,y_1 + \cdots + o^{1754}\,y_2 \\ g_{11} = o^{1386}\,x^3\,y_1^2 + x^3\,y_1\,y_2 + o^{2108}\,x^3\,y_1 + \cdots + o^{630}\,y_2 \\ \cdots \\ g_{19} = o^{3534}\,x^3\,y_1^2 + 41\,x^3\,y_1\,y_2 + o^{3210}\,x_3\,y_1 + \cdots + o^{1622}\,y_2 \end{cases}$$

Relations among $g_{10}, g_{11}, \cdots, g_{19}$:

$$\begin{cases} g_{11}^2 - (5g_{10}g_{12} + 42g_{10}g_{11} + 18g_{10}^2 + \cdots + 25) = 0 \\ g_{11}g_{12} - (26g_{10}g_{13} + 38g_{10}g_{12} + \cdots + 58) = 0 \\ \cdots \\ g_{12}g_{19} - (9g_{10}^2g_{11} + 62g_{10}^3 + 10g_{10}g_{19} + \cdots + 28) = 0 \end{cases}$$

↑ $C_{10,11,\ldots,19}$ curve over k in g_{10}-g_{11}-…-g_{19} space

Example1 : Reduction(1)



$E_w \ni G=(o^{387}\, r + o^{397},\ o^{166}\, r + o^{1205})$

From the definition of Π_2

$J_1 = \Pi_2^*(G)$

$= \{a((\beta - c)x+1)^2 - (G_x+\beta_2)((\beta - c^{q2})x+1)^2,$

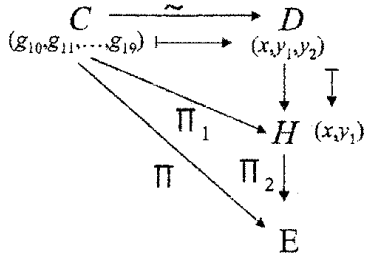
$a\alpha^{1/2} y_1 - G_y((\beta - c^{q2})x+1)^3\}$

$= \{(o^{353}r+o^{4196})x^2+(o^{1900}r+o^{1805})x+o^{1922}r+o^{2318},\ (o^{3720}r+o^{1533})x^3+$
$(o^{1693}r+o^{4323})x^2+(o^{3636}r+o^{1592})y_1+(o^{1256}r+o^{3701})x+o^{2686}r+o^{3725}\}$

To compute $\Pi_1^*(J)$, we use elimination ideal:

$J_2 \leftarrow \text{Eliminate}(J_1+(g_{10}-g_{10}(x,y_1,y_2), g_{11}-g_{11}(x,y_1,y_2),\ldots,$
$g_{19}-g_{19}(x,y_1,y_2),\ \{x,y_1,y_2\})$

Example1 : Reduction(2)



Finally we compute the norm:

$J = \Psi(G) = J_2 + J_2^{q} + J_2^{q2} + J_2^{q3}$

$= \{ g_{17}^2+37g_{17}+21g_{16}+49g_{15}+33g_{14}+\cdots+59,$
$g_{16}g_{17}+45g_{17}+15g_{16}+45g_{15}+21g_{14}+\cdots+63,$
$\cdots$
$g_{18}+24g_{17}+27g_{16}+31g_{15}+64g_{14}+\cdots+64\}$

Similarly, m=25415194-times point $G_m=(o^{637}r+o^{224},\ o^{1671}r+o^{3481})$ of G is transferred to

$J_m = \{ g_{17}^2+6g_{17}+70g_{16}+66g_{15}+15g_{14}+\cdots+68,$
$g_{16}g_{17}+5g_{17}+20g_{16}+56g_{15}+16g_{14}+\cdots+11,$
$\cdots$
$g_{18}+23g_{17}+34g_{16}+65g_{15}+18g_{14}+\cdots+4\}$

We verified $J_m = m \cdot J$ on C.

⇦ Gaudry method

Example2(160 bits length)

k: prime field of char. $p = 2^{40}-2^{35}-1$

k_2: quadratic ext. of k by $o^2 + 352619714346$

k_4: quartic ext. of k by $r^2 + 702753204573\ o + 465976829831$

E_w: elliptic curve over k_4

$v_1^2 = u_1^3+((773569929047o+698785454132)r+892468792697o+773390597884)u_1+$
$(245022657483o+657619174138)r+721187940068o+865450731541$

($\#E(k_4)$=1287200406650928609777376029597716043015507861907: 160 bits prime)

⇩

C: $C_{10,11,\ldots,19}$ curve over k

$$\begin{cases} g_{11}^2 - (671010913434\, g_{10} g_{12} + 306446345201\, g_{10} g_{11} + \\ \qquad 205461673669\, g_{10}^2 + \cdots + 675147796101) = 0 \\ g_{11} g_{12} - (752537421825\, g_{10} g_{13} + 1016531429604\, g_{10} g_{12} + \\ \qquad 897328181722\, g_{10} g_{11} + \cdots + 1053682994222) = 0 \\ \quad \cdots \\ g_{12} g_{19} - (128634052382\, g_{10}^2 g_{11} + 950367786029\, g_{10}^3 + \\ \qquad 457707828730\, g_{10} g_{19} + \cdots + 665817232135) = 0. \end{cases}$$

⇐ Gaudry method

Estimate of computational amounts

•Computational amounts of Gaudry method against C_{ab} curve of genus g over $GF(q)$ is

$$O(q^{2g/(g+1)+\varepsilon}) \quad (q \to \infty)$$

•So, computational amounts of our Weil descent attack(g=9):

$$q^{18/10} = q^{9/5} \ (< q^2) \quad (q \to \infty)$$

↑

Computational amounts of Pollard's rho method against elliptic curves on $GF(q^4)$:

More precise estimate

•Computational amounts of Gaudry method against C_{ab} curves of genus g defined on GF(q): ($l \geqq 1$: parameter)

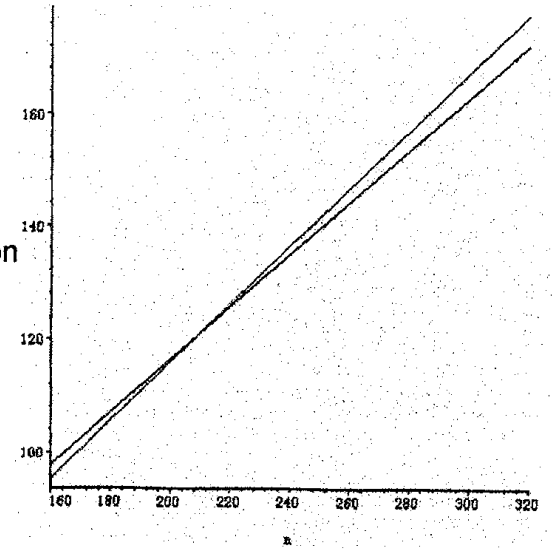
Minimum w.r.t. l of

$$l^{g-1} \cdot g^3 \cdot g! \cdot q \cdot (\log_2(q))^3 + l^{-2} \cdot g^3 \cdot q^2 \cdot (\log_2(q))^2$$

•Computational amounts of Pollard's rho method against elliptic curves on GF(q^4):

$$1.5 \cdot q^2 \cdot (\log_2(q^4))^2$$

Pollard v.s. Our Weil descent



——— Pollard's ρ method

——— Our Weil descent

intersection point:
210 bits

ρ method is stronger
in the real size parameters.